Invariant Differential Operators on Nonreductive Homogeneous Spaces

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Abstract

A systematic exposition is given of the theory of invariant differential operators on a not necessarily reductive homogeneous space. This exposition is modelled on Helgason's treatment of the general reductive case and the special nonreductive case of the space of horocycles. As a final application the differential operators on (not a priori reductive) isotropic pseudo-Riemannian spaces are characterized.

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1 Introduction

Let G be a Lie group and H a closed subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the corresponding Lie algebras. Suppose that the coset space G/H is *reductive*, i.e., there is a complementary subspace $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathrm{Ad}_G(H)\mathfrak{m} \subset \mathfrak{m}$. Let $\mathbb{D}(G/H)$ denote the algebra of G-invariant differential operators on G/H. The main facts about $\mathbb{D}(G/H)$ are summarized below (cf. HELGASON [3, Ch.III], [4, Cor. X.2.6, Theor. X.2.7], [6, §2]).

Let $\mathbb{D}(G)$ be the algebra of left invariant differential operators on G, $\mathbb{D}_H(G)$ the subalgebra of operators which are right invariant under H and $S(\mathfrak{g})$ the complexified symmetric algebra over $\mathfrak{g}$. Let $\lambda\colon S(\mathfrak{g}) \to \mathbb{D}(G)$ denote the symmetrization mapping. $I(\mathfrak{m})$ denotes the set of $\mathrm{Ad}_G(H)$-invariants in $S(\mathfrak{m})$. Then

$$\mathbb{D}_H(G) = \mathbb{D}(G)\mathfrak{h} \cap \mathbb{D}_H(G) \oplus \lambda(I(\mathfrak{m})). \tag{1.1}$$

Let $\pi\colon G \to G/H$ be the natural mapping. Let $C_H^\infty(G)$ consist of the C^∞-functions on G which are right invariant under H. Write $\tilde{f} := f \circ \pi$ ($f \in C^\infty(G/H)$) and $(D_u f)^\sim := u\tilde{f}$ ($f \in C^\infty(G/H)$, $u \in \mathbb{D}_H(G)$). Then $D_u \in \mathbb{D}(G/H)$.

Theorem 1.1 *The mapping $u \mapsto D_u$ is an algebra homomorphism from $\mathbb{D}_H(G)$ onto $\mathbb{D}(G/H)$ with kernel $\mathbb{D}(G)\mathfrak{h} \cap \mathbb{D}_H(G)$. The mapping $P \mapsto D_{\lambda(P)}\colon I(\mathfrak{m}) \to \mathbb{D}(G/H)$ is a linear bijection.*

Theorem 1.1. is of basic importance for the analysis on symmetric spaces. In particular, it can be shown that $\mathbb{D}(G/H)$ is commutative if G/H is a pseudo-Riemannian symmetric space which admits a relatively invariant measure. In its most general form this result was proved by DUFLO [1] in an algebraic way. G. van Dijk kindly communicated a short analytic proof of Duflo's result to me (unpublished). In [1] DUFLO used generalizations of (1.1) and Theorem 1.1 to the case of homogeneous line bundles over G/H. These can be proved by only minor changes of Helgason's original proofs.

There exist nonreductive coset spaces G/H for which $\mathbb{D}(G/H)$ is still commutative. For instance, let G be a connected real semisimple Lie group and let M and N be the usual subgroups of G. Then G/MN is the space of horocycles and $\mathbb{D}(G/MN)$ is commutative. In order to prove this, formula (1.1) and Theorem 1.1 have to be adapted to the nonreductive case. While HELGASON [5, §4], [6, §3] has done this in an ad hoc way for the special coset spaces under consideration, it is the purpose of the present note to give a more systematic exposition of the theory of $\mathbb{D}(G/H)$ for a not necessarily reductive coset space.

Furthermore, following Duflo, the theory will be developed for invariant differential operators on homogeneous line bundles over G/H. As a final application we will characterize $\mathbb{D}(G/H)$ for isotropic pseudo-Riemannian symmetric spaces G/H without a priori knowledge that G/H is reductive. Throughout HELGASON [4] will be our standard reference.

2 Development of the general theory

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$ define the vector field $\tilde{X}$ on G by

$$(\tilde{X}f)(g) := \frac{d}{dt} f(g \exp tX)\big|_{t=0}, \quad f \in C^\infty(G),\ g \in G. \tag{2.1}$$

Then the mapping $X \mapsto \tilde{X}$ is an isomorphism from $\mathfrak{g}$ onto the Lie algebra of left invariant vector fields on G. Throughout this section let $X_1, \ldots, X_n$ be a fixed basis of $\mathfrak{g}$.

For a finite-dimensional real vector space V the symmetric algebra $S(V)$ is defined as the algebra of all polynomials with complex coefficients on V^*, the dual of V. Let $S^m(V)$ respectively $S_m(V)$ ($m = 0, 1, 2, \ldots$) denote the space of homogeneous polynomials of degree m on V^*, respectively of polynomials of degree $\leq m$ on V^*. Thus $S^m(G)$ is spanned by the monomials $X_{i_1} X_{i_2} \ldots X_{i_m}$ ($i_1, \ldots, i_m \in \{1, \ldots, n\}$).

Let $\mathbb{D}(G)$ be the algebra of left invariant differential operators on G with complex coefficients. For $P \in S(\mathfrak{g})$ define an operator $\lambda(P)$ on $C^\infty(G)$ by

$$(\lambda(P)f)(g) := P\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}\right) f(g \exp(t_1 X_1 + \cdots + t_n X_n))\Big|_{t_1 = \cdots = t_n = 0}, \tag{2.2}$$

where

$$P\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}\right) := \frac{\partial^m}{\partial t_{i_1} \cdots \partial t_{i_m}} \quad \text{for } P = X_{i_1} \cdots X_{i_m}.$$

It is proved in [4, Prop. II.1.9 and p. 392] that:

Proposition 2.1 *The mapping $P \mapsto \lambda(P)$ is a linear bijection from $S(\mathfrak{g})$ onto $\mathbb{D}(G)$. It satisfies*

$$\lambda(Y^m) = \tilde{Y}^m, \quad Y \in \mathfrak{g}; \tag{2.3}$$

$$\lambda(Y_1 \dots Y_m) = \frac{1}{m!} \sum_{\sigma \in S_m} \tilde{Y}_{\sigma(1)} \dots \tilde{Y}_{\sigma(m)}, \quad Y_1, \dots, Y_m \in \mathfrak{g}. \tag{2.4}$$

The definition of λ is independent of the choice of the basis of $\mathfrak{g}$.

The mapping λ is called *symmetrization*. The Lie algebra $\mathfrak{g}$ is embedded as a subspace of $\mathbb{D}(G)$ under the mapping $X \to \tilde{X}$. Any homomorphism from $\mathfrak{g}$ to $\mathfrak{g}$ uniquely extends to a homomorphism from $\mathbb{D}(G)$ to $\mathbb{D}(G)$ and any linear mapping from $\mathfrak{g}$ to $\mathfrak{g}$ uniquely extends to a homorphism from $S(\mathfrak{g})$ to $S(\mathfrak{g})$. In particular, for $g \in G$, the automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g}$ uniquely extends to automorphisms of both $S(\mathfrak{g})$ and $\mathbb{D}(G)$ and

$$\lambda(\mathrm{Ad}(g)P) = \mathrm{Ad}(g)\lambda(P), \quad P \in S(\mathfrak{g}),\ g \in G. \tag{2.5}$$

For $g, g_1 \in G,\ f \in C^\infty(G),\ D \in \mathbb{D}(G)$ write

$$f^{R(g)}(g_1) := f(g_1 g); \quad D^{R(g)} f := (D f^{R(g^{-1})})^{R(g)}.$$

Then

$$\mathrm{Ad}(g)D = D^{R(g^{-1})}, \quad D \in \mathbb{D}(G),\ g \in G. \tag{2.6}$$

Let H be a closed subgroup of G and let $\mathfrak{h}$ be the corresponding subalgebra. Let $\mathfrak{m}$ be a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. Let $X_1, \dots, X_r$ be a basis of $\mathfrak{m}$ and $X_{r+1}, \dots, X_n$ a basis of $\mathfrak{h}$. Let χ be a character of H, i.e. a continuous homomorphism from H to the multiplicative group $\mathbb{C}\backslash\{0\}$. Throughout this section, H, $\mathfrak{m}$, the basis and χ will be assumed fixed.

Let $\pi\colon G \to G/H$ be the canonical mapping. Write $0 := \pi(e)$. Let

$$C^\infty_{H,\chi}(G) := \{f \in C^\infty(G) \mid f(gh) = f(g)\chi(h^{-1}),\ g \in G,\ h \in H\}. \tag{2.7}$$

Sometimes we will assume that χ has an extension to a character on G. This assumption clearly holds if $\chi \equiv 1$ on H, but it does not hold for general H and χ. For instance, if $G = SU(2)$ or $SL(2, \mathbb{R})$ and $H = SO(2)$ then nontrivial characters on H do not extend to characters on G.

If χ extends to a character on G then we define a linear bijection $f \mapsto \tilde{f}\colon C^\infty(G/H) \to C^\infty_{H,\chi}(G)$ by

$$\tilde{f}(g) := f(\pi(g))\chi(g^{-1}), \quad g \in G. \tag{2.8}$$

Lemma 2.2 *Let $P \in S(m)$. If $\lambda(P)f = 0$ for all $f \in C^\infty_{H,\chi}(G)$ then $P = 0$.*

Proof For each $f \in C^\infty(G/H)$ we can find $F \in C^\infty_{H,\chi}(G)$ such that

$$F(\exp(t_1 X_1 + \dots + t_r X_r)) = f(\exp(t_1 X_1 + \dots + t_r X_r) \cdot 0)$$

for $(t_1, \dots, t_r)$ in some neighbourhood of $(0, \dots, 0)$. Hence

$$0 = (\lambda(P)F)(e) = P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r}\right) f(\exp(t_1 X_1 + \dots + t_r X_r) \cdot 0\Big|_{t_1 = \dots = t_r = 0}$$

for all $f \in C^\infty(G/H)$, so $P = 0$. $\square$

Let the differential of χ also be denoted by χ. Let $\mathfrak{h}^{\mathbb{C}}$ be the complexification of $\mathfrak{h}$. Let

$$\mathfrak{h}^\chi := \{X + \chi(X) \mid X \in \mathfrak{h}^{\mathbb{C}}\} \subset \mathbb{D}(G). \tag{2.9}$$

Clearly, $Df = 0$ if $f \in C^\infty_{H,\chi}(G)$ and $D \in \mathfrak{h}^\chi$. Let $\mathbb{D}(G)\mathfrak{h}^\chi$ be the linear span of all vw with $v \in \mathbb{D}(G),\ w \in \mathfrak{h}^\chi$. Observe that, by Proposition 2.1, $\tilde{Y}_1 \dots \tilde{Y}_m \in \lambda(S_m(\mathfrak{g}))$ for $Y_1, \dots, Y_m \in \mathfrak{g}$. The following proposition was proved in [4, Lemma X.2.5] for $\chi \equiv 1$.

Proposition 2.3 *There are the direct sum decompositions*

$$\lambda(S_m(\mathfrak{g})) = \lambda(S_{m-1}(\mathfrak{g}))\mathfrak{h}^\chi \oplus \lambda(S_m(\mathfrak{m})) \tag{2.10}$$

and

$$\mathbb{D}(G) = \mathbb{D}(G)\mathfrak{h}^\chi \oplus \lambda(S(\mathfrak{m})). \tag{2.11}$$

Proof First we prove by complete induction with respect to $\mathfrak{m}$ that

$$\lambda(S_m(\mathfrak{g})) \subset \lambda(S_{m-1}(\mathfrak{g}))\mathfrak{h}^\chi + \lambda(S_m(\mathfrak{m})).$$

This clearly holds for $m = 0$. Suppose it is true for $m < d$. Let

$$P = X_1^{d_1} \cdots X_n^{d_n}, \quad d_1 + \cdots + d_n = d.$$

If $d_{r+1} \cdots + d_n = 0$, then $P \in S_d(\mathfrak{m})$, so $\lambda(P) \in \lambda(S_d(\mathfrak{m}))$. If $d_{r+1} + \cdots + d_n > 0$ then, by (2.4), $\lambda(P)$ is a linear combination of certain elements $\tilde{Y}_1 \cdots \tilde{Y}_d$ with $Y_i \in h$ for at least one i, so

$$\lambda(P) \in \lambda(S_{d-1}(\mathfrak{g}))\mathfrak{h}^{\mathbb{C}} + \lambda(S_{d-1}(\mathfrak{g})) \subset \lambda(S_{d-1}(\mathfrak{g}))\mathfrak{h}^\chi + \lambda(S_{d-1}(\mathfrak{g})).$$

Now apply the induction hypothesis. This yields (2.10) and (2.11) (use Proposition 2.1) except for the directness.

To prove the directness of the sum (2.11), suppose that $P \in S(\mathfrak{m})$ and $\lambda(P) \in \mathbb{D}(G)\mathfrak{h}^\chi$. Then $\lambda(P)f = 0$ for all $f \in C^\infty_{H,\chi}(G)$, so $P = 0$ by Lemma 2.2. $\square$

Lemma 2.4 *Let $D \in \mathbb{D}(G)$. Then $Df = 0$ for all $f \in C^\infty_{H,\chi}(G)$ if and only if $D \in \mathbb{D}(G)\mathfrak{h}^\chi$.*

Proof Apply Proposition 2.3 and Lemma 2.2. $\square$

Let us define

$$\mathbb{D}_{H,\chi,\mathrm{mod}}(G) := \{D \in \mathbb{D}(G) \mid \mathrm{Ad}(h)D - D \in \mathbb{D}(G)\mathfrak{h}^\chi \text{ for all } h \in H\}. \tag{2.12}$$

This definition is motivated by the following lemma.

Lemma 2.5 *Let $D \in \mathbb{D}(G)$. Then the following two statements are equivalent.*

(i) *$D \in \mathbb{D}_{H,\chi,\mathrm{mod}}(G)$.*

(ii) *$f \in C^\infty_{H,\chi}(G) \Rightarrow Df \in C^\infty_{H,\chi}(G)$.*

Proof Let $D \in \mathbb{D}(G)$. If $f \in C^\infty_{H,\chi}(G),\ h \in H$ then

$$(\star) \qquad (Df)^{R(h)} = D^{R(h)} f^{R(h)} = \chi(h^{-1}) D^{R(h)} f.$$

First assume (i). If $f \in C^\infty_{H,\chi}(G),\ h \in H$, then $(D^{R(h)} - D)f = (\mathrm{Ad}(h)D - D)f = 0$, so combination with $(\star)$ yields $(Df)^{R(h)} = \chi(h^{-1})Df$, i.e., $Df \in C^\infty_{H,\chi}(G)$. Conversely, assume (ii). If $f \in C^\infty_{H,\chi}(G),\ h \in H$, then $(Df)^{R(h)} = \chi(h^{-1})Df$, so combination with $(\star)$ yields $(D^{R(h)} - D)f = 0$. Hence $\mathrm{Ad}(h)D - D = D^{R(h)} - D \in \mathbb{D}(G)\mathfrak{h}^\chi$ by Lemma 2.4. □

From the preceding results the following theorem is now obvious.

Theorem 2.6

(a) $\mathbb{D}_{H,\chi,\mathrm{mod}}(G)$ *is a subalgebra of* $\mathbb{D}(G)$.

(b) $\mathbb{D}(G)\mathfrak{h}^\chi$ *is a two-sided ideal in* $\mathbb{D}_{H,\chi,\mathrm{mod}}(G)$.

(c) *There is the direct sum decomposition.*

$$\mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \mathbb{D}(G)\mathfrak{h}^\chi \oplus \lambda(S(\mathfrak{m})) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G). \tag{2.13}$$

(d) *Define the mappings A and B by*

$$\begin{array}{ccccc} u & \overset{A}{\longmapsto} & u(\mathrm{mod}\, \mathbb{D}(G)\mathfrak{h}^\chi) & \overset{B}{\longmapsto} & u|_{C^\infty_{H,\chi}(G)}: \\ \lambda(S(\mathfrak{m})) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G) & \overset{A}{\longrightarrow} & \mathbb{D}_{H,\chi,\mathrm{mod}}(G)/\mathbb{D}(G)\mathfrak{h}^\chi & \overset{B}{\longrightarrow} & \mathbb{D}_{H,\chi,\mathrm{mod}}\Big|_{C^\infty_{H,\chi}(G)}. \end{array}$$

Then A is a linear bijection and B is an algebra isomorphism onto.

Define the mapping $\sigma\colon \mathfrak{g} \to \mathfrak{m}$ by

$$\sigma(X + Y) := X, \quad X \in \mathfrak{m},\ Y \in \mathfrak{h}. \tag{2.14}$$

Consider $S(\mathfrak{m})$ as a subalgebra of $S(\mathfrak{g})$. Thus, if $P \in S(\mathfrak{m})$ and $h \in H$, then $\mathrm{Ad}(h)P \in S(\mathfrak{g})$ and $\sigma \circ \mathrm{Ad}(h)P \in S(\mathfrak{m})$ are well-defined. By an application of (2.4) we see that, if $Q \in S_m(\mathfrak{g})$, then

$$\lambda(\sigma Q - Q) \in \lambda(S_{m-1}(g)) + \mathbb{D}(G)\mathfrak{h}^\chi. \tag{2.15}$$

Define the algebra

$$I_{\mathrm{mod}}(\mathfrak{m}) := \{P \in S(\mathfrak{m}) \mid \sigma \circ \mathrm{Ad}(h)P = P \text{ for all } h \in H\}. \tag{2.16}$$

Lemma 2.7 *Let $P \in S(\mathfrak{m})$ such that $\lambda(P) \in \mathbb{D}_{H,\chi,\mathrm{mod}}(G)$. Write $P = P^m + P_{m-1}$, where $P^m \in S^m(\mathfrak{m}),\ P_{m-1} \in S_{m-1}(\mathfrak{m})$. Then $P^m \in I_{\mathrm{mod}}(\mathfrak{m})$.*

Proof $\lambda(\mathrm{Ad}(h)P - P) \in \mathbb{D}(G)\mathfrak{h}^\chi$ by (2.12). Hence

$$\lambda(\mathrm{Ad}(h)P^m - P^m) \in \lambda(S_{m-1}(\mathfrak{g})) + \mathbb{D}(G)\mathfrak{h}^\chi.$$

So

$$\lambda(\sigma \circ \mathrm{Ad}(h)P^m - P^m) \in \lambda(S_{m-1}(\mathfrak{g})) + \mathbb{D}(G)\mathfrak{h}^\chi \subset \lambda(S_{m-1}(\mathfrak{m})) + \mathbb{D}(G)\mathfrak{h}^\chi,$$

where we used (2.16) and (2.10). By directness of the decomposition (2.10):

$$\sigma \circ \mathrm{Ad}(h)P^m - P^m \in S_{m-1}(\mathfrak{m}).$$

Hence $\sigma \circ \mathrm{Ad}(h)P^m - P^m$, being homogeneous of degree m, is the zero polynomial. □

Proposition 2.8 *If* $\lambda(I_{\mathrm{mod}}(\mathfrak{m})) \subset \mathbb{D}_{H,\chi,\mathrm{mod}}(G)$ *then*

$$\lambda(I_{\mathrm{mod}}(\mathfrak{m})) = \lambda(S(\mathfrak{m})) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G)$$

and the mapping

$$D \mapsto D\Big|_{C^\infty_{H,\chi}(G)} : \lambda(I_{\mathrm{mod}}(\mathfrak{m})) \to \mathbb{D}_{H,\chi,\mathrm{mod}}(G)\Big|_{C^\infty_{H,\chi}(G)}$$

is a linear bijection.

Proof Use complete induction with respect to the degree of $P \in S(\mathfrak{m})$ in order to prove that $P \in I_{\mathrm{mod}}(\mathfrak{m})$ if $\lambda(P) \in \mathbb{D}_{H,\chi,\mathrm{mod}}(G)$ (apply Lemma 2.7). The second implication in the proposition follows from Theorem 2.6(d). □

Suppose for the moment that χ extends to a character on $\mathfrak{g}$ and remember the mapping $f \to \tilde{f}$ defined by (2.8). For $u \in \mathbb{D}_{H,\chi,\mathrm{mod}}(G)$ define an operator D_u acting on $C^\infty(G/H)$ by

$$(D_u f)^\sim := u\tilde{f}, \quad f \in C^\infty(G/H). \tag{2.17}$$

Then $\mathrm{supp}(D_u f) \subset \mathrm{supp}(f)$, hence, by Peetre's theorem (cf. for instance NARASIMHAN [7, §3.3]), D_u is a differential operator on G/H. One easily shows that $D_u \in \mathbb{D}(G/H)$, the space of G-invariant differential operators on G/H.

Theorem 2.9 *Suppose that* χ *extends to a character on* G*. Then the mapping*

$$u\Big|_{C^\infty_{H,\chi}(G)} \stackrel{C}{\longmapsto} D_u\colon \quad \mathbb{D}_{H,\chi,\mathrm{mod}}(G)\Big|_{C^\infty_{H,\chi}(G)} \stackrel{C}{\longrightarrow} \mathbb{D}(G/H)$$

is an algebraic isomorphism onto.

Proof Clearly, C is an isomorphism into. In order to prove the surjectivity let $D \in \mathbb{D}(G/H)$. Then there is a polynomial $P \in S(\mathfrak{m})$ such that

$$(Df)(g \cdot 0) = P\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_r}\right) f(g \exp(t_1 X_1 + \cdots + t_r X_r) \cdot 0\Big|_{t_1 = \cdots = t_r = 0}$$

for all $f \in C^\infty(G/H)$ and for $g = e$. By the G-invariance of D this formula holds for all $g \in G$. By (2.8) and (2.2) this becomes

$$\chi(Df)^\sim = \lambda(P)(\chi\tilde{f}), \quad \text{i.e.,} \quad (Df)^\sim = (\chi^{-1}\lambda(P) \circ \chi)(\tilde{f}).$$

Clearly, $\chi^{-1}\lambda(P) \circ \chi \in \mathbb{D}(G)$ and, by Lemma 2.5, we have $\chi^{-1}\lambda(P) \circ \chi \in \mathbb{D}_{H,\chi\mathrm{mod}}(G)$. Thus, by (2.17), $D = D_{\chi^{-1}\lambda(P)\circ\chi}$. □

Suppose now that the coset space G/H is *reductive*, i.e., $\mathfrak{m}$ can be chosen such that $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$. From now on assume that $\mathfrak{m}$ is chosen in this way. Let

$$\begin{aligned} \mathbb{D}_H(G) &:= \{D \in \mathbb{D}(G) \mid \mathrm{Ad}(h)D = D \quad \text{for all } h \in H\}, & (2.18)\\ I(\mathfrak{m}) &:= \{P \in S(\mathfrak{m}) \mid \mathrm{Ad}(h)P = P \quad \text{for all } h \in H\}. & (2.19) \end{aligned}$$

Then

$$\lambda(S(\mathfrak{m})) \cap \mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \lambda(I(\mathfrak{m})) \subset \mathbb{D}_H(G).$$

Hence (2.13) becomes

$$\mathbb{D}_{H,\chi,\mathrm{mod}}(G) = \mathbb{D}(G)\mathfrak{h}^\chi \oplus \lambda(I(\mathfrak{m})). \tag{2.20}$$

We obtain from Theorems 2.6 and 2.9:

Theorem 2.10 *Let G/H be reductive. Then:*

(a) $\mathbb{D}_H(G)$ *is a subalgebra of* $\mathbb{D}(G)$.

(b) $\mathbb{D}(G)\mathfrak{h}^\chi \cap \mathbb{D}_H(G)$ *is a two-sided ideal in* $\mathbb{D}_H(G)$.

(c) *There is a direct sum decomposition*

$$\mathbb{D}_H(G) = \mathbb{D}(G)\mathfrak{h}^\chi \cap \mathbb{D}_H(G) \oplus \lambda(I(\mathfrak{m})). \tag{2.21}$$

(d) *Define the mappings A, B and C (C only if χ extends to a character on G) by*

$$\begin{array}{rcl} u & \overset{A}{\longmapsto} & u(\mathrm{mod}\,\mathbb{D}(G)\mathfrak{h}^\chi \cap \mathbb{D}_H(G)) \overset{B}{\longmapsto} u\Big|_{C^\infty_{H,\chi}(G)} \overset{C}{\longmapsto} D_u: \\ \lambda(I(\mathfrak{m})) & \overset{A}{\longrightarrow} & \mathbb{D}_H(G)/(\mathbb{D}(G)\mathfrak{h}^\chi \cap \mathbb{D}_H(G)) \overset{B}{\longrightarrow} \mathbb{D}_H(G)\Big|_{C^\infty_{H,\chi}(G)} \overset{C}{\longrightarrow} \mathbb{D}(G/H). \end{array}$$

Then A is a linear bijection and B and C are algebra isomorphisms onto.

The case $\chi \equiv 1$ of Theorem 2.10 can be found in HELGASON [4, Cor. X.2.6 and Theor. X.2.7]. See DUFLO [1] for the general case.

3 Application to $\mathbb{D}(G/N)$ and $\mathbb{D}(G/MN)$

Let G be a connected noncompact real semisimple Lie group. We remember some of the structure theory of G (cf. [3. Ch.VI]):

$\mathfrak{g}_0$: Lie algebra of G.
$\mathfrak{g}$: complexification of $\mathfrak{g}_0$.
θ : Cartan involution of $\mathfrak{g}_0$, extended to automorphism of $\mathfrak{g}$.
$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$: corresponding Cartan decomposition of $\mathfrak{g}_0$.
$\mathfrak{h}_{\mathfrak{p}_0}$: mamximal abelian subspace of $\mathfrak{p}_0$, A the coresponding analytic subgroup.
$\mathfrak{h}_0$: maximal abelian subalgebra of $\mathfrak{g}_0$ extending $\mathfrak{h}_{\mathfrak{p}_0}$.
$\mathfrak{h}_{\mathfrak{k}_0} := \mathfrak{h}_0 \cap \mathfrak{k}_0$, $\quad \mathfrak{h}_{\mathfrak{k}}$ its complexification
$\mathfrak{h}$: complexification of $\mathfrak{h}_0$; this is a Cartan subalgebra of $\mathfrak{g}$.
Δ : set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$; the roots are real on $i\mathfrak{h}_{\mathfrak{k}_0} + \mathfrak{h}_{\mathfrak{p}_0}$.

Introduce compatible orderings on $\mathfrak{h}_{\mathfrak{p}_0}^*$ and $(i\mathfrak{h}_{\mathfrak{k}_0} + \mathfrak{h}_{\mathfrak{p}_0})^*$.

Δ^+ : set of positive roots.
P_+ : set of positive roots not vanishing on $\mathfrak{h}_{\mathfrak{p}_0}$.
P_- : set of positive roots vanishing on $\mathfrak{h}_{\mathfrak{p}_0}$.
$\mathfrak{g}^\alpha$: root space in $\mathfrak{g}$ of $\alpha \in \Delta$.
$\mathfrak{n}$: $\sum_{\alpha \in P_+} \mathfrak{g}^\alpha$.
$\mathfrak{n}_0 := \mathfrak{n} \cap \mathfrak{g}_0$.
N : analytic subgroup of G corresponding to $\mathfrak{n}_0$.
M : centralizer of $\mathfrak{h}_{\mathfrak{p}_0}$ in G, M_0 its identity component.
$\mathfrak{m}_0$: Lie algebra of M.
$\mathfrak{m}$: complexification of $\mathfrak{m}_0$; then

$$\mathfrak{m} = \mathfrak{h}_{\mathfrak{k}} + \sum_{\alpha \in P_-} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}). \tag{3.1}$$

Proposition 3.1 *The coset spaces G/MN and G/N are not reductive.*

Proof Suppose that G/MN is reductive. Then there is an $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{m}+\mathfrak{n})$-invariant subspace $\mathfrak{r}$ of $\mathfrak{g}$ complementary to $\mathfrak{m}+\mathfrak{n}$. Let $\alpha \in P_+$ and let X be a nonzero element of $\mathfrak{g}^\alpha$. For $H \in \mathfrak{h}$ write $H = W_H + Y_H + Z_H$ with $W_H \in \mathfrak{r}$, $Y_H \in \mathfrak{m}$, $Z_H \in \mathfrak{n}$. Then, for each $H \in \mathfrak{h}$:

$$\alpha(H)X = [W_H + Y_H + Z_H, X]$$

so

$$\alpha(H)X - [Y_H, X] - [Z_H, X] = [W_H, X] \in \mathfrak{r} \cap (\mathfrak{m} + \mathfrak{n}),$$

so

$$[Y_H, X] + [Z_H, X] = \alpha(H)X.$$

It follows from (3.1) that

$$[Y_H, X] + [Z_H, X] \in \sum_{\substack{\beta \in \Delta \\ \beta \neq \alpha}} \mathfrak{g}^\beta.$$

Hence $\alpha(H)X = 0$ for all $H \in h$, so $\alpha = 0$. This is a contradiction.

In the case G/N the proof is almost the same: take $\mathfrak{r}$ $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})$-invariant and complementary to $\mathfrak{n}$ and $Y_H = 0$. □

HELGASON [5, p. 676] states without proof that G/MN is not in general reductive.

Let $\mathfrak{l}_0$ be the orthoplement of $\mathfrak{m}_0$ in $\mathfrak{k}_0$ with respect to the Killing form on $\mathfrak{g}_0$. In order to apply Proposition 2.8 and Theorem 2.9 to $\mathbb{D}(G/MN)$ and $\mathbb{D}(G/N)$ we take $\mathfrak{l}_0 + \mathfrak{h}_{\mathfrak{p}_0}$ respectively $\mathfrak{k}_0 + \mathfrak{h}_{\mathfrak{p}_0}$ as complementary subspaces of $\mathfrak{m}_0 + \mathfrak{n}_0$ respectively $\mathfrak{n}_0$ in $\mathfrak{g}_0$. Now we have

$$I_{\mathrm{mod}}(\mathfrak{l}_0 + \mathfrak{h}_{\mathfrak{p}_0}) = S(\mathfrak{h}_{\mathfrak{p}_0}), \tag{3.2}$$

$$I_{\mathrm{mod}}(\mathfrak{k}_0 + \mathfrak{h}_{\mathfrak{p}_0}) = S(\mathfrak{m}_0 + \mathfrak{h}_{\mathfrak{p}_0}). \tag{3.3}$$

(3.2) is proved in HELGASON [5. Lemma 4.2] and by only slight modifications in this proof, (3.3) is obtained. It follows from Lemma 2.5 that

$$\lambda(S(\mathfrak{h}_{\mathfrak{p}_0})) \subset \mathbb{D}_{MN,1,\mathrm{mod}}(G)$$

and

$$\lambda(S(\mathfrak{m}_0 + \mathfrak{h}_{\mathfrak{p}_0})) \subset \mathbb{D}_{N,1,\mathrm{mod}}(G),$$

since M centralizes $\mathfrak{h}_{\mathfrak{p}_0}$ and $\mathfrak{m}_0 + \mathfrak{h}_{\mathfrak{p}_0}$ normalizes $\mathfrak{n}_0$. Consider $\mathbb{D}(A)$ and $\mathbb{D}(M_0A)$ as subalgebras of $\mathbb{D}(G)$. Then $\mathbb{D}(A) \subset \mathbb{D}_{MN,1,\mathrm{mod}}(G)$ and $\mathbb{D}(M_0A) \subset \mathbb{D}_{N,1,\mathrm{mod}}(G)$. It follows by application of Proposition 2.8 and Theorem 2.9 that:

Theorem 3.2 *The mapping $u \mapsto D_u$ (cf. (2.17)) is an algebra isomorphism from $\mathbb{D}(A)$ onto $\mathbb{D}(G/MN)$ and from $\mathbb{D}(M_0A)$ onto $\mathbb{D}(G/N)$. In particular, $\mathbb{D}(G/MN)$ is a commutative algebra.*

The statements about $\mathbb{D}(G/MN)$ are in HELGASON [5, Theorem 4.1]. FARAUT [2, p. 393] observes that Helgason's result can be extended to the context of pseudo-Riemannian symmetric spaces.

A special case of Theorem 6.2 can be formulated in the situation that G is a connected complex semisimple Lie group. Let $\mathfrak{g}$ be its (complex) Lie algebra and put:

$\mathfrak{u}$: compact real form of $\mathfrak{g}$.
$\mathfrak{a}$: maximal abelian subalgebra of $\mathfrak{u}$.
$\mathfrak{h} := \mathfrak{a} + i\mathfrak{a}$; this is Cartan subalgebra of $\mathfrak{g}$.
Δ : set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.
Δ^+ : set of positive roots with respect to some ordering.
$\mathfrak{g}^\alpha$: root space of $\alpha \in \Delta$.
$\mathfrak{n} := \sum_{\alpha\in\Delta^+} \mathfrak{g}^\alpha$, N the corresponding analytic subgroup.
$\mathfrak{g}^{\mathbb{R}} := \mathfrak{g}$ considered as real Lie algebra.
$\mathfrak{h}^{\mathbb{R}} := \mathfrak{h}$ considered as real subalgebra.

Then $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} + i\mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition for $\mathfrak{g}^{\mathbb{R}}$ (cf. [4, Theorem VI.6.3]) and $\mathfrak{a}$ is the centralizer of $i\mathfrak{a}$ in $\mathfrak{u}$. Hence we obtain from Theorem 3.2:

Theorem 3.3 *The mapping $P \mapsto D_{\lambda(P)}$ is an algebra isomorphism from $S(\mathfrak{h}^{\mathbb{R}})$ onto $\mathbb{D}(G/N)$. In particular, $\mathbb{D}(G/N)$ is commutative.*

This theorem was proved by HELGASON [6, Lemma 3.3] without use of Theorem 3.2.

4 Application to isotropic spaces

We preserve the notation and conventions of Section 2. First we prove an extension of [4, Cor. X.2.8] to the case that G/H is not necessarily reductive. In the following, A and B are as in Theorem 2.6(d).

Lemma 4.1 *If the algebra $I_{\rm mod}(\mathfrak{m})$ is generated by $P_1, \ldots, P_l$ and if there are $Q_1, \ldots, Q_l \in S_m$ such that* $\text{degree}(P_i - Q_i) < \text{degree}\, P_i$ *and $\lambda(Q_i) \in \mathbb{D}_{H,\chi,\rm mod}(G)$ then the algebra*

$$\mathbb{D}_{H,\chi,\rm mod}\Big|_{C^\infty_{H,\chi}(G)}$$

is generated by $BA\lambda(Q_1), \ldots, BA\lambda(Q_l)$.

Proof We prove by complete induction with respect to m that, for each $P \in S_m(\mathfrak{m})$ with $\lambda(P) \in \mathbb{D}_{H,\chi,\rm mod}(G)$, $BA\lambda(P)$ depends polynomially on $BA\lambda(Q_1), \ldots, BA\lambda(Q_l)$. In view of Theorem 2.6 this will prove the lemma. Suppose the above property holds up to $m-1$. Let $P \in S_m(\mathfrak{m})$ such that $\lambda(P) \in \mathbb{D}_{H,\chi,\rm mod}(G)$. By using Lemma 2.7 we find that $P = \Pi(P_1, \ldots, P_l) \pmod{S_{m-1}(\mathfrak{m})}$ for some polynomial Π in l indeterminates. Hence, $P = \Pi(Q_1, \ldots, Q_l) \pmod{S_{m-1}(\mathfrak{m})}$,

$$\begin{aligned}
\lambda(P) &= \lambda(\Pi(Q_1, \ldots, Q_l)) \pmod{\lambda(S_{m-1}(\mathfrak{m}))} \\
&= \Pi(\lambda(Q_1), \ldots, \lambda(Q_l)) \pmod{\lambda(S_{m-1}(\mathfrak{g}))}, \\
\lambda(P) &- \Pi(\lambda(Q_1), \ldots, \lambda(Q_l)) \in \lambda(S_{m-1}(\mathfrak{g})) \cap \mathbb{D}_{H,\chi,\rm mod}(G).
\end{aligned}$$

By Theorem 2.6 and formula (2.10) we have

$$BA\lambda(P) - \Pi(BA\lambda(Q_1), \ldots, BA\lambda(Q_l)) = BA\lambda(P')$$

for some $P' \in S_{m-1}(\mathfrak{m})$ such that $\lambda(P') \in \mathbb{D}_{H,\chi,\rm mod}(G)$. Now apply the induction hypothesis. □

Let τ denote the action of G on G/H. Its differential $d\tau$ yields an action of H on the tangent space $(G/H)_0$ to G/H at 0.

Theorem 4.2 *Suppose there is a nondegenerate $d\tau(H)$-invariant bilinear form $\langle\cdot,\cdot\rangle$ on $(G/H)_0$ of signature (r_1, r_2) $(r_1 + r_2 = r,\ r_1 \geq r_2)$ such that, for each $\lambda > 0$, $d\tau(H)$ acts transitively on $\{X \in (G/H)_0 \mid \langle X, X\rangle = \lambda\}$ (or on the connected components of these hyperbolas if $r_1 = r_2 = 1$). Let Δ be the Laplace-Beltrami operator on G/H coresponding to the G-invariant pseudo-Riemannian structure on G/H associated with $\langle\cdot,\cdot\rangle$. Then the algebra $\mathbb{D}(G/H)$ is generated by Δ, and hence commutative.*

Proof Choose a complementary subspace $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$. The mapping $d\pi$ identifies the H-spaces $\mathfrak{m}$ (under $\sigma \circ \mathrm{Ad}_G(H)$) and $(G/H)_0$ (under $d\tau(H)$) with each other. Transplant the form $\langle\cdot,\cdot\rangle$ to $\mathfrak{m}$ and choose an orthonormal basis $X_1, \ldots, X_r$ of $\mathfrak{m}$: $\langle X_i, X_j\rangle = \varepsilon_i \delta_{ij}$, $\varepsilon_i = 1$ or -1 for $i \leq r_1$ or $> r_1$, respectively. Then the algebra $I_{\rm mod}(\mathfrak{m})$ is generated by $\sum_{u=1}^r \varepsilon_i X_i^2$.

It follows from the proof of Theorem 2.9 that $\Delta = D_{\lambda(P)}$ with $P \in S(\mathfrak{m})$ of degree 2 such that $\lambda(P) \in \mathbb{D}_{H,1,\mathrm{mod}}(G)$. Thus, by Lemma 2.7, we get

$$P = c \sum_{i=1}^{r} \varepsilon_i X_i^2 \pmod{S_1(\mathfrak{m})}$$

with $c \neq 0$. Now apply Lemma 4.1 and Theorem 2.9. $\square$

Theorem 4.2 extends [4, Prop. X.2.10], where the case is considered that G/H is a Riemannian symmetric space of rank 1. A pseudo-Riemannian manifold M is called *isotropic* if for each $x \in M$ and for tangent vectors $X, Y \neq 0$ at x with $\langle X, X \rangle = \langle Y, Y \rangle$ there is an isometry of M fixing x which sends X to Y. Connected isotropic spaces can be written as homogeneous spaces G/H satisfying the conditions of Theorem 4.2 with G being the full isometry group (cf. WOLF [8, Lemma 11.6.6]). It follows from Wolf's classification [8, Theorem 12.4.5] that such spaces are symmetric and reductive. However, our proof of Theorem 4.2 does not use this fact.

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